

# The construction of characteristic matrixes of dynamic coverings using an incremental approach

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**Abstract.** The covering approximation space evolves in time due to the explosion of the information, and the characteristic matrixes of coverings viewed as an effective approach to approximating the concept should update with time for knowledge discovery. This paper further investigates the construction of characteristic matrixes without running the matrix acquisition algorithm repeatedly. First, we present two approaches to computing the characteristic matrixes of the covering with lower time complexity. Then, we investigate the construction of the characteristic matrixes of the dynamic covering using the incremental approach. We mainly address the characteristic matrix updating from three aspects: the variations of elements in the covering, the immigration and emigration of objects and the changes of attribute values. Afterwards, several illustrative examples are employed to show that the proposed approach can effectively compute the characteristic matrixes of the dynamic covering for approximations of concepts.

**Keywords:** Rough sets; Dynamic covering; Boolean matrix; Characteristic matrix

## 1 Introduction

Covering-based rough set theory [26], as a powerful mathematical tool for studying information systems, has attracted a lot of attention in recent years. Actually, the covering rough sets is regarded as a meaningful extension of Pawlak's model [11–14] to deal with more complex data sets. Nowadays it [2–6, 15, 17–25, 28, 29] has been successfully applied to pattern recognition, machine learning and environmental science because of its approximation ability.

Constructing the approximations of concepts is one major work of the covering-based rough set theory by using reasonable approximation operators. On one hand, the literatures [23, 24, 30–38] have already

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provided several models of covering-based rough sets and multiple fuzzy rough set models based on fuzzy coverings. On the other hand, researchers studied the basic properties of the proposed models of covering-based rough sets. For example, Mordeson [10] discussed the algebraic structural properties of certain subsets for a type of covering-based rough sets. Wang et al. [17] investigated the data compression of the covering information system. Yang et al. [23] unified the reduction of different types of covering generalized rough sets. Zhu and Wang [30–37] studied five types of covering-based rough sets systematically. Naturally, it motivates us to further study the covering-based rough set theory.

Recently, Wang et al. [16] represented and axiomatized three types of covering approximation operators by using two types of characteristic matrixes. In other words, the computation of the approximation of a set is transformed into the product of the characteristic matrix of the covering and the characteristic function of the set. The results presents a new view to discuss the covering-based rough sets by borrowing extensively from boolean matrixes. Actually, one major work is to construct the two types of characteristic matrixes of the covering in the process of computing the approximations of concepts. But Wang et al. [16] paid little attention to the approach to constructing the characteristic matrixes, and it is of interest to introduce an effective approach to computing the characteristic matrixes with low time complexity. On the other hand, the covering approximation space varies with time due to the characteristics of data collection in practice, and the non-incremental approach to constructing the approximations of concepts in the dynamic covering approximation space is often very costly or even intractable. As we know, some scholars [1, 7–9, 27] studied attribute reductions of the dynamic information systems. Therefore, it is interesting to apply an incremental updating scheme to maintain the approximations of sets dynamically and avoid unnecessary computations by utilizing the approximations in the original covering approximation space.

The purpose of this paper is to further study the approximations of concepts using the characteristic matrixes. First, we introduce two approaches to approximating the concept using the characteristic matrixes of the covering. More concretely, we can compress the covering approximation space into a small-scale one under the condition of the consistent function and obtain the approximation of the concept by computing the approximation of its image. Thus the characteristic matrixes of the covering can be transformed into a small one, and it can reduce the time complexity of constructing the approximation of the concept. Subsequently, we present another approach to computing the characteristic matrixes of the covering by constructing the matrix representation of each element in the covering, and it is useful for the construction of the characteristic matrixes of the dynamic covering. Especially, we can apply the two approaches to constructing the approximation of the concept simultaneously. Second, we show that how to get the characteristic matrixes of the dynamic covering by using an incremental approach. We mainly focus on five types of dynamic coverings: adding elements into the covering and deleting the elements of the covering, the immigration and emigration of the object sets, and revising the attribute values of some objects. We investigate the relationship between the characteristic matrixes of the original and dynamic coverings. Several examples are employed to illustrate that how to update the characteristic matrixes of

the dynamic coverings by utilizing an incremental approach.

The rest of this paper is organized as follows: Section 2 briefly reviews the basic concepts of the covering information systems. In Section 3, we present two approaches to computing the characteristic matrixes of the covering. Section 4 is devoted to constructing the type-1 and type-2 characteristic matrixes of the dynamic covering by utilizing the incremental approach. We conclude the paper in Section 5.

## 2 Preliminaries

In this section, we review some concepts of the covering and characteristic matrixes. In addition, we investigate the basic properties of the characteristic matrixes.

**Definition 2.1** [26] *Let  $U$  be a finite universe of discourse, and  $\mathcal{C}$  a family of subsets of  $U$ . If  $\mathcal{C}$  satisfies the conditions:*

- (1) *none of elements of  $\mathcal{C}$  is empty;*
- (2)  *$\bigcup\{C \mid C \in \mathcal{C}\} = U$ , then  $\mathcal{C}$  is called a covering of  $U$ .*

It is obvious that the concept of the covering is an extension of the partition. In addition,  $(U, \mathcal{C})$  is called a covering approximation space if  $\mathcal{C}$  is a covering of the universe  $U$ .

To compress the covering approximation space, Wang et al. [17] provided the concept of consistent functions with respect to the covering.

**Definition 2.2** [17] *Let  $f$  be a mapping from  $U_1$  to  $U_2$ ,  $\mathcal{C} = \{C_1, C_2, \dots, C_N\}$  a covering of  $U_1$ ,  $N(x) = \bigcap \{C_i \mid x \in C_i \text{ and } C_i \in \mathcal{C}\}$ , and  $[x]_f = \{y \in U_1 \mid f(x) = f(y)\}$ . If  $[x]_f \subseteq N(x)$  for any  $x \in U_1$ , then  $f$  is called a consistent function with respect to  $\mathcal{C}$ .*

Based on Definition 2.2, Wang et al. constructed a homomorphism between a complex massive covering information system and a relatively small-scale covering information system, and the homomorphism provides the foundation for the communication between covering information systems.

Recently, Wang et al. [16] introduced the concepts of characteristic matrixes for constructing the approximations of concepts.

**Definition 2.3** [16] *Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe,  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a family of subsets of  $U$ , and  $M_{\mathcal{C}} = (c_{ij})_{n \times m}$ , where  $c_{ij} = \begin{cases} 1, & x_i \in C_j; \\ 0, & x_i \notin C_j. \end{cases}$  Then  $M_{\mathcal{C}}$  is called a matrix representation of  $\mathcal{C}$ .*

In the sense of Definition 2.3, if  $\mathcal{C}$  is a covering of the universe  $U$ , then  $M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T$  is called the type-1 characteristic matrix of  $\mathcal{C}$ , denoted as  $\Gamma(\mathcal{C})$ , where  $M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T$  is the boolean product of  $M_{\mathcal{C}}$  and its transpose  $M_{\mathcal{C}}^T$ . Furthermore,  $\Gamma(\mathcal{C})$  is the relational matrix of the relation induced by indiscernible neighborhoods of the covering, denoted as  $R_{\mathcal{C}}$ . Especially, we have that  $(x, y) \in R_{\mathcal{C}}$  iff  $y \in I(x) = \{K \in \mathcal{C} \mid x \in K\}$  for all  $x, y \in U$ .

**Proposition 2.4** Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe,  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a family of subsets of  $U$ , and  $M_C \cdot M_C^T = (c_{ij})_{n \times n}$  the type-1 characteristic matrix of  $C \in \mathcal{C}$ . Then we have that  $c_{ii} = 1$  iff  $x_i \in C$ .

**Proof.** Suppose that  $M_C = [a_1, a_2, \dots, a_n]^T$  is the matrix representation of  $C \in \mathcal{C}$ . In the sense of Definition 2.3, we have that  $a_i = 1$  if  $x_i \in C$ . It follows that  $a_i \wedge a_i = 1$ . Thus,  $c_{ii} = 1$ .

By Definition 2.3, we have that  $c_{ij} = a_i \wedge a_j$ . It implies that  $a_i = 1$ . Therefore,  $x_i \in C$ .  $\square$

**Proposition 2.5** Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe,  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a family of subsets of  $U$ , and  $M_A \cdot M_A^T = (a_{ij})_{n \times n}$ ,  $M_B \cdot M_B^T = (b_{ij})_{n \times n}$ ,  $M_C \cdot M_C^T = (c_{ij})_{n \times n}$  the type-1 characteristic matrixes of  $A, B, C \in \mathcal{C}$ , respectively. Then we have that  $c_{ii} = a_{ii} \vee b_{ii}$  iff  $C$  is the union of  $A$  and  $B$ .

**Proof.** By Proposition 2.4, if  $c_{ii} = 1$ , then we have that  $a_{ii} = 1$  or  $b_{ii} = 1$ . It implies that  $x_i \in A$  or  $x_i \in B$  if  $x_i \in C$ . Furthermore, if  $c_{ii} = 0$ , then  $a_{ii} = 0$  and  $b_{ii} = 0$ . In other words,  $x_i \notin A$  and  $x_i \notin B$  if  $x_i \notin C$ . Thus  $C$  is the union of  $A$  and  $B$ .

If  $C$  is the union of  $A$  and  $B$ , then that  $x_i \in C$  implies that  $x_i \in A$  or  $x_i \in B$ . It follows that  $c_{ii} = 1$  if  $a_{ii} = 1$  or  $b_{ii} = 1$ . Thus  $c_{ii} = a_{ii} \vee b_{ii}$ . Moreover, that  $x_i \notin C$  implies that  $x_i \notin A$  and  $x_i \notin B$ . Then we have that  $a_{ii} = 0$ ,  $b_{ii} = 0$  and  $c_{ii} = 0$ . It follows that  $c_{ii} = a_{ii} \vee b_{ii}$ . Therefore,  $c_{ii} = a_{ii} \vee b_{ii}$  if  $C$  is the union of  $A$  and  $B$ .  $\square$

By Proposition 2.5, we have that  $c_{ii} = a_{ii} \vee b_{ii}$  iff  $C$  is the union of  $A$  and  $B$ . An example is employed to illustrate that  $M_C \cdot M_C^T = (a_{ij})_{n \times n} = M_A \cdot M_A^T \vee M_B \cdot M_B^T$  does not necessarily hold if  $C$  is the union of  $A$  and  $B$ .

**Example 2.6** Let  $U = \{x_1, x_2, x_3, x_4\}$ ,  $A = \{x_1, x_2\}$ ,  $B = \{x_1, x_4\}$  and  $C = \{x_1, x_2, x_4\}$ . By Definition 2.3, we have that  $M_A \cdot M_A^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $M_B \cdot M_B^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ , and  $M_C \cdot M_C^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ . It is obvious that  $C = A \cup B$ . But  $M_C \cdot M_C^T \neq M_A \cdot M_A^T \vee M_B \cdot M_B^T$ .

**Proposition 2.7** Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe,  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a family of subsets of  $U$ , and  $M_A \cdot M_A^T = (a_{ij})_{n \times n}$ ,  $M_C \cdot M_C^T = (b_{ij})_{n \times n}$ ,  $M_C \cdot M_C^T = (c_{ij})_{n \times n}$  the type-1 characteristic matrixes of  $A, B$  and  $C \in \mathcal{C}$ , respectively. Then

- (1)  $C$  is the intersection of  $A$  and  $B$  iff  $c_{ii} = a_{ii} \wedge b_{ii}$ ;
- (2)  $C$  is the intersection of  $A$  and  $B$  iff  $M_C \cdot M_C^T = M_A \cdot M_A^T \wedge M_B \cdot M_B^T$ .

**Proof.** (1) The proof is similar to that of Proposition 2.5.

(2) By Proposition 2.7 (1), the proof is straightforward.  $\square$

**Definition 2.8** [16] Let  $A = (a_{ij})_{n \times m}$  and  $B = (b_{ij})_{m \times p}$  be two boolean matrixes, and  $C = A \odot B = (c_{ij})_{n \times p}$ .

Then  $c_{ij}$  is defined as

$$c_{ij} = \bigwedge_{k=1}^m (b_{kj} - a_{ik} + 1).$$

In the sense of Definition 2.8, if  $\mathcal{C}$  is a covering of the universe of  $U$ , then  $M_{\mathcal{C}} \odot M_{\mathcal{C}}^T$  is called the type-2 characteristic matrix of  $\mathcal{C}$ , denoted as  $\Pi(\mathcal{C})$ . Furthermore,  $\Pi(\mathcal{C})$  is the relational matrix of the relation induced by neighborhoods of the covering, denoted as  $R(\mathcal{C})$ . Specially, we have that  $(x, y) \in R(\mathcal{C})$  iff  $y \in N(x)$  for all  $x, y \in U$ .

**Definition 2.9** [16] Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe, and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a covering of  $U$ . For any  $X \subseteq U$ , the second, fifth and sixth upper and lower approximations of  $X$  with respect to  $\mathcal{C}$ , respectively, are defined as follows:

- (1)  $SH_{\mathcal{C}}(X) = \bigcup \{C \in \mathcal{C} \mid C \cap X \neq \emptyset\}$ ,  $SL_{\mathcal{C}}(X) = [SH_{\mathcal{C}}(X^c)]^c$ ;
- (2)  $IH_{\mathcal{C}}(X) = \bigcup \{x \in U \mid N(x) \cap X \neq \emptyset\}$ ,  $IL_{\mathcal{C}}(X) = \bigcup \{x \in U \mid N(x) \subseteq X\}$ ;
- (3)  $XH_{\mathcal{C}}(X) = \bigcup \{N(x) \in U \mid N(x) \cap X \neq \emptyset\}$ ,  $XL_{\mathcal{C}}(X) = \bigcup \{N(x) \in U \mid N(x) \subseteq X\}$ , where  $N(x) = \bigcap \{K \in \mathcal{C} \mid x \in K\}$ .

Throughout the paper, we delete  $\mathcal{C}$  in the description of the lower and upper approximation operators for simplicity. By using the type-1 and type-2 characteristic matrixes, Wang et al. represented equivalently and axiomatized three important types of covering approximation operators.

**Definition 2.10** [16] Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe, and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a covering of  $U$ . Then

- (1)  $\mathcal{X}_{SH(X)} = \Gamma(\mathcal{C}) \cdot \mathcal{X}_X$ ,  $\mathcal{X}_{SL(X)} = \Gamma(\mathcal{C}) \odot \mathcal{X}_X$ ;
- (2)  $\mathcal{X}_{IH(X)} = \Pi(\mathcal{C}) \cdot \mathcal{X}_X$ ,  $\mathcal{X}_{IL(X)} = \Pi(\mathcal{C}) \odot \mathcal{X}_X$ ;
- (3)  $\mathcal{X}_{XH(X)} = \Pi(\mathcal{C})^T \cdot \Pi(\mathcal{C}) \cdot \mathcal{X}_X$ ,  $\mathcal{X}_{XL(X)} = \Pi(\mathcal{C})^T \cdot \Pi(\mathcal{C}) \odot \mathcal{X}_X$ , where  $\mathcal{X}_X$  denotes the characteristic function of  $X$  in  $U$ . In other words, for any  $y \in U$ ,  $\mathcal{X}_X(x) = 1$  iff  $x \in X$ .

**Proposition 2.11** Let  $U = \{x_1, x_2, \dots, x_n\}$  be a finite universe, and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  a covering of  $U$ . Then  $\Pi(\mathcal{C}) \leq \Gamma(\mathcal{C})$  and  $\Pi(\mathcal{C})^T \cdot \Pi(\mathcal{C}) = \Pi(\mathcal{C})$ .

**Proof.** For all  $x, y \in U$ , if we have that  $y \in N(x)$ , then it implies that  $y \in I(x)$ . But the converse does not hold necessarily. It follows that  $R_{\mathcal{C}} \subseteq R(\mathcal{C})$ . Thus  $\Pi(\mathcal{C}) \leq \Gamma(\mathcal{C})$ .

Since  $IH_{\mathcal{C}}(X) = XH_{\mathcal{C}}(X)$ , we obtain that  $\mathcal{X}_{IH(X)} = \mathcal{X}_{XH(X)}$ . Therefore,  $\Pi(\mathcal{C})^T \cdot \Pi(\mathcal{C}) = \Pi(\mathcal{C})$ .  $\square$

### 3 Two approaches to constructing approximations of concepts using characteristic matrixes

In this section, we introduce two approaches to constructing the approximation of the concept by using the characteristic matrixes, and the proposed approaches can be applied to the dynamic covering

approximation space.

Suppose that  $(U, \mathcal{C})$  is a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$  is a finite universe, and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  is a covering of  $U$ . In the sense of Definitions 2.3 and 2.8, we see that  $\Pi(\mathcal{C})$  and  $\Gamma(\mathcal{C})$  are  $n \times n$  matrixes if the number of the objects of the universe is  $n$ . Obviously, the time complexity of computing the approximation of the concept by using the approach [16] is high if there are a large number of objects in the universe. To solve this issue, we introduce an approach to constructing the approximation of the concept by using the consistent function. There are two steps for computing the approximation of the concept. First, we compress the covering approximation space  $(U, \mathcal{C})$  into a relative small scale one  $(f(U), f(\mathcal{C}))$  by using the consistent function  $f$ . Then we obtain the approximation  $f^{-1}(Y)$  of  $X$  in  $(U, \mathcal{C})$  by constructing the approximation  $Y$  of  $f(X)$  in the  $(f(U), f(\mathcal{C}))$ .

An example is employed to illustrate the process of computing approximations of concepts by using the proposed approach.

**Example 3.1** Let  $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ ,  $\mathcal{C} = \{C_1, C_2, C_3\}$ ,  $C_1 = \{x_1, x_2\}$ ,  $C_2 = \{x_3, x_4, x_5, x_6\}$ ,  $C_3 = \{x_1, x_2, x_5, x_6\}$ , and  $X = \{x_1, x_2, x_3, x_4\}$ . Then

$$\Gamma(\mathcal{C}) = M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

First, we get the upper and lower approximations of  $X$  by computing

$$\mathcal{X}_{SH(X)} = \Gamma(\mathcal{C}) \cdot \mathcal{X}_X = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathcal{X}_{SL(X)} = \Gamma(\mathcal{C}) \odot \mathcal{X}_X = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Second, we construct the consistent function  $f$  from  $U$  to  $V$  as follows:

$$f(x_1) = f(x_2) = y_1, f(x_3) = f(x_4) = y_2, f(x_5) = f(x_6) = y_3,$$

and get  $(V, f(\mathcal{C}))$ , where  $V = \{y_1, y_2, y_3\}$ ,  $f(\mathcal{C}) = \{f(C_1), f(C_2), f(C_3)\}$ ,  $f(C_1) = \{y_1\}$ ,  $f(C_2) = \{y_2, y_3\}$ , and  $f(C_3) = \{y_1, y_3\}$ . Thus we can compress  $X$  into  $f(X) = \{y_1, y_2\}$  and get the lower and upper approximations of  $f(X)$  by computing  $\mathcal{X}_{SH(f(X))} = \Gamma(f(\mathcal{C})) \cdot \mathcal{X}_{f(X)}$  and  $\mathcal{X}_{SL(f(X))} = \Gamma(f(\mathcal{C})) \odot \mathcal{X}_{f(X)}$  as

follows:

$$\begin{aligned}\Gamma(f(\mathcal{C})) &= M_{f(\mathcal{C})} \cdot M_{f(\mathcal{C})}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \\ \mathcal{X}_{SH(f(X))} &= \Gamma(f(\mathcal{C})) \cdot \mathcal{X}_{f(X)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \\ \mathcal{X}_{SL(f(X))} &= \Gamma(\mathcal{C}) \odot \mathcal{X}_{f(X)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.\end{aligned}$$

It is obvious that  $SH(f(X)) = \{y_1, y_2, y_3\}$  and  $SL(f(X)) = \{y_1\}$ . Therefore,  $SH(X) = f^{-1}(SH(f(X))) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$  and  $SL(X) = f^{-1}(SL(f(X))) = \{x_1, x_2\}$ .

Subsequently, we introduce another approach to constructing the approximation of the concept by using the characteristic matrixes. In the sense of Definition 2.3, we have that  $M_C = (d_{i1})_{n \times 1}$  for any  $C \in \mathcal{C}$ , where  $d_{i1} = \begin{cases} 1, & x_i \in C; \\ 0, & x_i \notin C \end{cases}$ , and  $M_C$  is called the matrix representation of  $C$ . Actually, we can obtain  $M_C \cdot M_C^T = (c_{ij})_{n \times n}$  by identifying the value of  $d_{i1}$ . Concretely, we have that  $c_{ik} = d_{k1}$  ( $1 \leq k \leq n$ ) if  $d_{i1} = 1$ . Otherwise, we have that  $c_{ik} = 0$  for  $1 \leq k \leq n$  if  $d_{i1} = 0$ . Thereby, we only need to identify  $d_{i1} = 1$  or  $d_{i1} = 0$ .

We employ the following example to show the process of computing  $M_C \cdot M_C^T$  with the proposed approach.

**Example 3.2** Let  $U = \{x_1, x_2, x_3, x_4\}$ ,  $C = \{x_1, x_4\}$ , and  $M_C = (d_{ij})_{4 \times 1} = [1 \ 0 \ 0 \ 1]^T$ . Then

$$M_C \cdot M_C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot [1 \ 0 \ 0 \ 1] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

From the above results, we see that  $d_{11} = d_{41} = 1$  and  $d_{21} = d_{31} = 0$ . It is obvious that  $c_{1j} = c_{4j} = d_{j1}$  and  $c_{2j} = c_{3j} = 0$  ( $1 \leq j \leq 4$ ). Therefore, we can get  $M_C \cdot M_C^T$  without computing  $c_{ij}$  ( $1 \leq i, j \leq 4$ ), and the time complexity of computing  $M_C \cdot M_C^T$  can be reduced greatly.

Then we investigate the construction of the type-1 characteristic matrix based on Example 3.2. For the covering approximation space  $(U, \mathcal{C})$ , we obtain the matrix representations  $\{M_{C_i} \cdot M_{C_i}^T | C_i \in \mathcal{C}\}$  shown in Table 1. Since Wang et al. have proved that  $M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T = \bigvee_{C_i \in \mathcal{C}} M_{C_i} \cdot M_{C_i}^T$ , we can get  $M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T$  by computing  $M_{C_i} \cdot M_{C_i}^T$ . Especially,  $\{M_{C_i} \cdot M_{C_i}^T | C_i \in \mathcal{C}\}$  are useful for computing the type-1 characteristic matrix of the dynamic covering, which are illustrated in Section 4.

Table 1: The type-1 characteristic matrix of each  $C_i \in \mathcal{C}$  and  $\mathcal{C}$ .

$U$	$C_1$	$C_2$	.	.	.	$C_m$	$\mathcal{C}$
$M_C \cdot M_C^T$	$M_{C_1} \cdot M_{C_1}^T$	$M_{C_2} \cdot M_{C_2}^T$	.	.	.	$M_{C_m} \cdot M_{C_m}^T$	$M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T$

The following example is employed to illustrate the computing of the type-1 characteristic matrix with the proposed approach.

**Example 3.3** Let  $U = \{x_1, x_2, x_3, x_4\}$ ,  $\mathcal{C} = \{C_1, C_2, C_3\}$ , where  $C_1 = \{x_1, x_4\}$ ,  $C_2 = \{x_1, x_2, x_4\}$ , and  $C_3 = \{x_3, x_4\}$ . Then we have that

$$M_{C_1} \cdot M_{C_1}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, M_{C_2} \cdot M_{C_2}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{C_3} \cdot M_{C_3}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, we obtain that

$$M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T = M_{C_1} \cdot M_{C_1}^T \vee M_{C_2} \cdot M_{C_2}^T \vee M_{C_3} \cdot M_{C_3}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Additionally, we discuss that how to get  $M_C \odot M_C^T$  with low time complexity. Actually, we can obtain  $M_C \odot M_C^T$  by computing two rows of it, denoted as  $(c_{ij})_{n \times n}$ . Concretely, we have that  $c_{ik} = c_{jk}$  ( $1 \leq k \leq n$ ) if  $d_{i1} = d_{j1}$ . So we only need to compute  $c_{ik}$  and  $c_{jk}$  ( $1 \leq k \leq n$ ) when  $d_{i1} = 1$  and  $d_{j1} = 0$ , respectively.

We employ an example to show the process of computing  $M_C \odot M_C^T$  using the proposed approach.

**Example 3.4** (Continuation of Example 3.2) Using the proposed approach, we have that

$$M_C \odot M_C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

From the above results, we see that  $d_{11} = d_{41} = 1$  and  $d_{21} = d_{31} = 0$ . It is obvious that  $c_{1j} = c_{4j}$  and  $c_{2j} = c_{3j}$  ( $1 \leq j \leq 4$ ). To get  $M_C \odot M_C^T$ , we can only compute  $c_{1j}$  and  $c_{2j}$  ( $1 \leq j \leq 4$ ). In this way, the time complexity of computing  $M_C \odot M_C^T$  can be reduced greatly.

After that, we study the construction of the type-2 characteristic matrix of the covering based on Example 3.4. Concretely, we obtain  $\{M_{C_i} \odot M_{C_i}^T | C_i \in \mathcal{C}\}$  shown in Table 2. In the sense of Definition 2.8, we have that  $M_{\mathcal{C}} \odot M_{\mathcal{C}}^T = \bigwedge_{C_i \in \mathcal{C}} M_{C_i} \odot M_{C_i}^T$ . Thus we can get  $M_{\mathcal{C}} \odot M_{\mathcal{C}}^T$  by computing  $M_{C_i} \odot M_{C_i}^T$ . Furthermore,  $\{M_{C_i} \odot M_{C_i}^T | C_i \in \mathcal{C}\}$  are useful for computing the type-2 characteristic matrix of the dynamic covering, which are illustrated in Section 4.

Table 2: The matrix representations of each  $C_i \in \mathcal{C}$  and  $\mathcal{C}$ .

$U$	$C_1$	$C_2$	.	.	.	$C_m$	$\mathcal{C}$
$M_C \odot M_C^T$	$M_{C_1} \odot M_{C_1}^T$	$M_{C_2} \odot M_{C_2}^T$	.	.	.	$M_{C_m} \odot M_{C_m}^T$	$M_{\mathcal{C}} \odot M_{\mathcal{C}}^T$

We employ the following example to show the process of computing the type-2 characteristic matrix with the proposed approach.



**Example 3.5** (Continuation of Example 3.3) Similarly, we have that

$$M_{C_1} \odot M_{C_1}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}, M_{C_2} \odot M_{C_2}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{C_3} \odot M_{C_3}^T = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, we obtain that

$$M_{\mathcal{C}} \odot M_{\mathcal{C}}^T = M_{C_1} \odot M_{C_1}^T \wedge M_{C_2} \odot M_{C_2}^T \wedge M_{C_3} \odot M_{C_3}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is obvious that the time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $m * O(n) + O(n^2)$  if  $|U| = n$  and  $|\mathcal{C}| = m$ . But the time complexity is  $O(m * n^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix. Especially, we can compute  $M_{C_i} \cdot M_{C_i}^T$  and  $M_{C_j} \cdot M_{C_j}^T$  for  $i \neq j$  simultaneously. Thus the time complexity can be reduced to  $O(n) + O(n^2)$ . Therefore, we can get the type-1 and type-2 characteristic matrixes with less time by using the proposed approach.

In fact, the proposed approaches can be applied to compute approximations of concepts simultaneously. Concretely, we can compress the covering approximation space into a small one before using the second approach.

## 4 The construction of characteristic matrixes of the dynamic covering

In this section, we investigate that how to update the type-1 and type-2 characteristic matrixes with time. Actually, there exists five types of dynamic coverings: adding elements into the covering and deleting some elements of the covering, the immigration and emigration of object sets, and revising attribute values of some objects.

### 4.1 The characteristic matrixes of the dynamic covering when varying elements of the covering

In the following, we introduce the concept of the dynamic covering approximation space when adding some elements and investigate the type-1 and type-2 characteristic matrixes of the dynamic covering when adding some elements into the covering.

**Definition 4.1** Let  $(U, \mathcal{C})$  be a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ , and  $\mathcal{C}^* = \{C_i^* | m+1 \leq i \leq k\}$ . Then  $(U, \mathcal{C})$  is called the original covering approximation space and  $(U, \mathcal{C}^+)$  is called the AE-covering approximation space of  $(U, \mathcal{C})$ , where  $\mathcal{C}^+ = \mathcal{C} \cup \mathcal{C}^*$ .

In the sense of Definition 4.1,  $\mathcal{C}$  is called the original covering and  $\mathcal{C}^+$  is called the AE-covering of the original covering. Obviously, we can obtain the type-1 and type-2 characteristic matrixes of the AE-covering as the original covering.

We discuss that how to get the type-1 characteristic matrix of  $\mathcal{C}^+$  based on that of  $\mathcal{C}$ . On one hand, it is obvious that  $M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T = M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T \vee M_{\mathcal{C}^*} \cdot M_{\mathcal{C}^*}^T$ , so we can obtain the result by computing  $M_{\mathcal{C}^*} \cdot M_{\mathcal{C}^*}^T$ . But this approach is ineffective for future computing if some elements of  $\mathcal{C}^+$  are deleted. On the other hand, we compute and add  $\{M_{C_i^*} \cdot M_{C_i^*}^T | C_i^* \in \mathcal{C}^*\}$  into Table 1, and the results are shown in Table 3. In the process of computing  $M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T$ , there is no need to compute  $\{M_{C_i} \cdot M_{C_i}^T | C_i \in \mathcal{C}\}$  by using the results of the original covering.

Table 3: The matrix representation of each element of the covering  $\mathcal{C}^+$ .

$U$	$C_1$	$C_2$	.	.	.	$C_k$	$\mathcal{C}^+$
$M_C \cdot M_C^T$	$M_{C_1} \cdot M_{C_1}^T$	$M_{C_2} \cdot M_{C_2}^T$	.	.	.	$M_{C_k} \cdot M_{C_k}^T$	$M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T$

The following example is employed to show that how to compute the type-1 characteristic matrix of the AE-covering with the proposed approach.

**Example 4.2** (Continuation of Example 3.3) Let  $\mathcal{C}^+ = \mathcal{C} \cup \{C_4\}$ , where  $C_4 = \{x_2, x_4\}$ . Obviously,  $\mathcal{C}^+$  is the AE-covering of  $\mathcal{C}$ . To compute  $M_{\mathcal{C}^+}$ , we only need to compute  $M_{C_4} \cdot M_{C_4}^T$  as follows:

$$M_{C_4} \cdot M_{C_4}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Consequently, we obtain that

$$M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T = M_{C_1} \cdot M_{C_1}^T \vee M_{C_2} \cdot M_{C_2}^T \vee M_{C_3} \cdot M_{C_3}^T \vee M_{C_4} \cdot M_{C_4}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Furthermore, we can get

$$M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T = M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T \vee M_{C_4} \cdot M_{C_4}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We also investigate that how to construct the type-2 characteristic matrix of  $\mathcal{C}^+$  based on that of  $\mathcal{C}$ . It is obvious that  $M_{\mathcal{C}^+} \odot M_{\mathcal{C}^+}^T = M_{\mathcal{C}} \odot M_{\mathcal{C}}^T \wedge M_{\mathcal{C}^*} \odot M_{\mathcal{C}^*}^T$ , so we can obtain the result by computing  $M_{\mathcal{C}^*} \odot M_{\mathcal{C}^*}^T$ . But this approach is also ineffective for future computing if some elements of  $\mathcal{C}^+$  are deleted. Therefore, we focus on the approach shown in Section 3. Concretely, we compute  $\{M_{C_i^*} \odot M_{C_i^*}^T | C_i^* \in \mathcal{C}^*\}$  and show the results in Table 4.

Table 4: The matrix representation of each element of the covering  $\mathcal{C}^+$ .

$U$	$C_1$	$C_2$	.	.	.	$C_k$	$\mathcal{C}^+$
$M_C \odot M_C^T$	$M_{C_1} \odot M_{C_1}^T$	$M_{C_2} \odot M_{C_2}^T$	.	.	.	$M_{C_k} \odot M_{C_k}^T$	$M_{\mathcal{C}^+} \odot M_{\mathcal{C}^+}^T$

We employ the following example to show the process of computing the type-2 characteristic matrix of the dynamic covering with the proposed approach.

**Example 4.3** (Continuation of Example 4.2) To get  $M_{\mathcal{C}^+} \odot M_{\mathcal{C}^+}^T$ , we only need to compute  $M_{C_4} \odot M_{C_4}^T$  as follows:

$$M_{C_4} \odot M_{C_4}^T = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then we obtain that

$$M_{\mathcal{C}^+} \odot M_{\mathcal{C}^+}^T = M_{C_1} \odot M_{C_1}^T \wedge M_{C_2} \odot M_{C_2}^T \wedge M_{C_3} \odot M_{C_3}^T \wedge M_{C_4} \odot M_{C_4}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

On the other hand, we can get

$$M_{\mathcal{C}^+} \odot M_{\mathcal{C}^+}^T = M_{\mathcal{C}} \odot M_{\mathcal{C}}^T \wedge M_{C_4} \odot M_{C_4}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $(k - m) * O(n) + O(n^2)$  if  $|U| = n$  and  $|\mathcal{C}^*| = k$ . But the time complexity is  $O(k * n^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix. Especially, we can compute  $M_{C_i} \cdot M_{C_i}^T$  and  $M_{C_j} \cdot M_{C_j}^T$  for  $m \leq i \neq j \leq k$  simultaneously. Thus we can get the type-1 and type-2 characteristic matrixes with less time by using the proposed approach.

Subsequently, we introduce the concept of the dynamic covering approximation space when deleting some elements of the covering and investigate the type-1 and type-2 characteristic matrixes of the dynamic covering when deleting some elements of the covering.

**Definition 4.4** Let  $(U, \mathcal{C})$  be a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$ ,  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ , and  $\mathcal{C}^* \subseteq \mathcal{C}$ . Then  $(U, \mathcal{C}^-)$  is called the DE-covering approximation space of  $(U, \mathcal{C})$ , where  $\mathcal{C}^- = \mathcal{C} - \mathcal{C}^*$ .

In the sense of Definition 4.4,  $\mathcal{C}$  is called the original covering and  $\mathcal{C}^-$  is called the DE-covering of the original covering. Furthermore, we can obtain the type-1 and type-2 characteristic matrixes of the DE-covering as the original covering.

We discuss that how to get the type-1 and type-2 characteristic matrixes of  $\mathcal{C}^-$  based on those of  $\mathcal{C}$ . There are two steps to compute the type-1 and type-2 characteristic matrixes of  $\mathcal{C}^-$ . To express clearly, we assume that only  $C_N \in \mathcal{C}$  is deleted. First, we get Tables 5 and 6 by deleting  $M_{C_N} \cdot M_{C_N}^T$  and  $M_{C_N} \odot M_{C_N}^T$  in Tables 2 and 3, respectively. Second, we obtain that  $M_{\mathcal{C}^-} \cdot M_{\mathcal{C}^-}^T = \bigvee_{C_i \in \mathcal{C}^-} M_{C_i} \cdot M_{C_i}^T$  and  $M_{\mathcal{C}^-} \odot M_{\mathcal{C}^-}^T = \bigwedge_{C_i \in \mathcal{C}^-} M_{C_i} \odot M_{C_i}^T$ . Clearly, there is no need to compute  $\{M_{C_i} \cdot M_{C_i}^T | C_i \in \mathcal{C} - \{C_N\}\}$  and  $\{M_{C_i} \odot M_{C_i}^T | C_i \in \mathcal{C} - \{C_N\}\}$ . Similarly, we can get the type-1 and type-2 characteristic matrixes of the DE-covering when a set of elements of the original covering is deleted simultaneously.

Table 5: The matrix representation of each element of the covering  $\mathcal{C}^-$ .

$U$	$C_1$	$C_2$	$\dots$	$C_{N-1}$	$C_{N+1}$	$\dots$	$C_m$	$\mathcal{C}^-$
$M_C \cdot M_C^T$	$M_{C_1} \cdot M_{C_1}^T$	$M_{C_2} \cdot M_{C_2}^T$	$\dots$	$M_{C_{N-1}} \cdot M_{C_{N-1}}^T$	$M_{C_{N+1}} \cdot M_{C_{N+1}}^T$	$\dots$	$M_{C_m} \cdot M_{C_m}^T$	$M_{\mathcal{C}^-} \cdot M_{\mathcal{C}^-}^T$

 Table 6: The matrix representation of each element of the covering  $\mathcal{C}^-$ .

$U$	$C_1$	$C_2$	$\dots$	$C_{N-1}$	$C_{N+1}$	$\dots$	$C_m$	$\mathcal{C}^-$
$M_C \odot M_C^T$	$M_{C_1} \odot M_{C_1}^T$	$M_{C_2} \odot M_{C_2}^T$	$\dots$	$M_{C_{N-1}} \odot M_{C_{N-1}}^T$	$M_{C_{N+1}} \odot M_{C_{N+1}}^T$	$\dots$	$M_{C_m} \odot M_{C_m}^T$	$M_{\mathcal{C}^-} \odot M_{\mathcal{C}^-}^T$

The following example is employed to show that how to compute the type-1 and type-2 characteristic matrixes of the DE-covering with the proposed approach.

**Example 4.5** (Continuation of Example 3.3) Let  $\mathcal{C}^- = \mathcal{C} - \{C_3\}$ , and  $\mathcal{C}^-$  is the DE-covering of  $\mathcal{C}$ . Then we have that

$$\begin{aligned}
 M_{\mathcal{C}^-} \cdot M_{\mathcal{C}^-}^T &= M_{C_1} \cdot M_{C_1}^T \vee M_{C_2} \cdot M_{C_2}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}; \\
 M_{\mathcal{C}^-} \odot M_{\mathcal{C}^-}^T &= M_{C_1} \odot M_{C_1}^T \wedge M_{C_2} \odot M_{C_2}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

In the process of computing  $M_{\mathcal{C}^-} \cdot M_{\mathcal{C}^-}^T$  and  $M_{\mathcal{C}^-} \odot M_{\mathcal{C}^-}^T$ , there is no need to compute  $M_{C_1} \cdot M_{C_1}^T$ ,  $M_{C_2} \cdot M_{C_2}^T$ ,  $M_{C_1} \odot M_{C_1}^T$  and  $M_{C_2} \odot M_{C_2}^T$ . Thereby, the computing of the type-1 and type-2 characteristic matrixes of the DE-covering can be reduced greatly.

The time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $O(n^2)$  if  $|U| = n$  and  $|\mathcal{C}^-| = k$ . But the time complexity is  $O(k * n^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix.

## 4.2 The characteristic matrixes of the dynamic covering when adding some new objects and deleting some objects

In the following, we introduce the concept of the dynamic covering approximation space when adding some objects and investigate the type-1 and type-2 characteristic matrixes of the dynamic covering when adding some objects.

**Definition 4.6** Let  $(U, \mathcal{C})$  be a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ ,  $U^* = \{x_i | n+1 \leq i \leq n+t\}$ ,  $U^+ = U \cup U^*$ ,  $\mathcal{C}^+ = \{C_1^+, C_2^+, \dots, C_m^+\}$ ,  $C_i^+$  is updated by adding some new objects into  $C_i$ . Then  $(U^+, \mathcal{C}^+)$  is called the AO-covering approximation space of  $(U, \mathcal{C})$ .

In the sense of Definition 4.6,  $\mathcal{C}$  is called the original covering and  $\mathcal{C}^+$  is called the updated covering of the original covering. Furthermore, we can obtain the type-1 and type-2 characteristic matrixes of the updated covering as the original covering.

We discuss that how to get the type-1 characteristic matrix of  $\mathcal{C}^+$  based on that of  $\mathcal{C}$ . Concretely, we study the relationship between  $\Gamma(\mathcal{C}) = (b_{ij})_{(n+t) \times (n+t)}$  and  $\Gamma(\mathcal{C}^+) = (c_{ij})_{(n+t) \times (n+t)}$ . It is obvious that  $\Gamma(\mathcal{C}^+)$  is symmetric and  $b_{ij} = c_{ij}$  ( $1 \leq i, j \leq n$ ). Thus we only need to compute  $c_{ij}$  ( $n+1 \leq i \leq n+t, 1 \leq j \leq n+t$ ), denoted as  $\Delta\Gamma(\mathcal{C})$ . Therefore, the time complexity of computing the type-1 characteristic matrix of  $\mathcal{C}^+$  can be reduced greatly.

$$\begin{aligned}
\Gamma(\mathcal{C}) = M_{\mathcal{C}} \cdot M_{\mathcal{C}}^T &= \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix}^T \\
&= \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nm} \end{bmatrix}; \\
\Gamma(\mathcal{C}^+) = M_{\mathcal{C}^+} \cdot M_{\mathcal{C}^+}^T &= \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \\ a_{(n+1)1} & a_{(n+1)2} & \cdot & \cdot & \cdot & a_{(n+1)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n+t)1} & a_{(n+t)2} & \cdot & \cdot & \cdot & a_{(n+t)m} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \\ a_{(n+1)1} & a_{(n+1)2} & \cdot & \cdot & \cdot & a_{(n+1)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n+t)1} & a_{(n+t)2} & \cdot & \cdot & \cdot & a_{(n+t)m} \end{bmatrix}^T \\
&= \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} & c_{1(n+1)} & \cdot & \cdot & \cdot & c_{1(n+t)} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} & c_{2(n+1)} & \cdot & \cdot & \cdot & c_{2(n+t)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdot & \cdot & \cdot & c_{nn} & c_{n(n+1)} & \cdot & \cdot & \cdot & c_{n(n+t)} \\ c_{(n+1)1} & c_{(n+1)2} & \cdot & \cdot & \cdot & c_{(n+1)n} & c_{(n+1)(n+1)} & \cdot & \cdot & \cdot & c_{(n+1)(n+t)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{(n+t)1} & c_{(n+t)2} & \cdot & \cdot & \cdot & c_{(n+t)n} & c_{(n+t)(n+1)} & \cdot & \cdot & \cdot & c_{(n+t)(n+t)} \end{bmatrix}; \\
\Delta\Gamma(\mathcal{C}) &= \begin{bmatrix} c_{(n+1)1} & c_{(n+1)2} & \cdot & \cdot & \cdot & c_{(n+1)n} & c_{(n+1)(n+1)} & \cdot & \cdot & \cdot & c_{(n+1)(n+t)} \\ c_{(n+2)1} & c_{(n+2)2} & \cdot & \cdot & \cdot & c_{(n+2)n} & c_{(n+2)(n+1)} & \cdot & \cdot & \cdot & c_{(n+2)(n+t)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{(n+t)1} & c_{(n+t)2} & \cdot & \cdot & \cdot & c_{(n+t)n} & c_{(n+t)(n+1)} & \cdot & \cdot & \cdot & c_{(n+t)(n+t)} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} a_{(n+1)1} & a_{(n+1)2} & \cdot & \cdot & \cdot & a_{(n+1)m} \\ a_{(n+2)1} & a_{(n+2)2} & \cdot & \cdot & \cdot & a_{(n+2)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n+t)1} & a_{(n+t)2} & \cdot & \cdot & \cdot & a_{(n+t)m} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \\ a_{(n+1)1} & a_{(n+1)2} & \cdot & \cdot & \cdot & a_{(n+1)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n+t)1} & a_{(n+t)2} & \cdot & \cdot & \cdot & a_{(n+t)m} \end{bmatrix}^T.$$

The following example is employed to show that how to compute the type-1 characteristic matrix of the dynamic covering with the proposed approach.

**Example 4.7** (Continuation of Example 3.3) Let  $U^+ = U \cup \{x_5, x_6\}$ ,  $\mathcal{C}^+ = \{C_1^+, C_2^+, C_3^+\}$ , where  $C_1^+ = \{x_1, x_4, x_5\}$ ,  $C_2^+ = \{x_1, x_2, x_4, x_5, x_6\}$ , and  $C_3^+ = \{x_3, x_4, x_6\}$ , and  $\mathcal{C}^-$  is the AO-covering of  $\mathcal{C}$ . Then we obtain that

$$\Delta\Gamma(\mathcal{C}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we obtain  $\Gamma(\mathcal{C}^+) = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$

Then, we discuss that how to get the type-2 characteristic matrix of  $\mathcal{C}^+$  based on that of  $\mathcal{C}$ . We study the relationship between  $\prod(\mathcal{C}) = (b_{ij})_{(n+t) \times (n+t)}$  and  $\prod(\mathcal{C}^+) = (c_{ij})_{(n+t) \times (n+t)}$ . It is obvious that  $b_{ij} = c_{ij}$  ( $1 \leq i, j \leq n$ ). Thus we only need to compute  $\Delta_1 \prod = (c_{ij})_{(n+1 \leq i \leq n+t, 1 \leq j \leq n+t)}$  and  $\Delta_2 \prod = (c_{ij})_{(1 \leq i \leq n, n+1 \leq j \leq n+t)}$ . Therefore, the time complexity of computing the type-2 characteristic matrix of  $\mathcal{C}^+$  can be reduced greatly.

$$\begin{aligned} \prod(\mathcal{C}) = M_{\mathcal{C}} \odot M_{\mathcal{C}}^T &= \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix} \odot \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix}^T \\ &= \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nn} \end{bmatrix}; \end{aligned}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix} \odot \begin{bmatrix} a_{(n+1)1} & a_{(n+1)2} & \cdot & \cdot & \cdot & a_{(n+1)m} \\ a_{(n+2)1} & a_{(n+2)2} & \cdot & \cdot & \cdot & a_{(n+2)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n+t)1} & a_{(n+t)2} & \cdot & \cdot & \cdot & a_{(n+t)m} \end{bmatrix}^T$$

The following example is employed to show that how to compute the type-1 characteristic matrix of the dynamic covering with the proposed approach.

**Example 4.8** (Continuation of Example 4.7) By Definition 4.6,  $\mathcal{C}^-$  is the AO-covering of  $\mathcal{C}$ . Thus we obtain that

$$\begin{aligned}\Delta_1 \prod(\mathcal{C}) &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}; \\ \Delta_2 \prod(\mathcal{C}) &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Therefore, we obtain  $\prod(\mathcal{C}^+) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$

The time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $O((n+t)^2) + O(m * t * (n+t))$  if  $|U^*| = n+t$  and  $|\mathcal{C}| = m$ . But the time complexity is  $O(m * (n+t)^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix. Therefore, we can get the type-1 and type-2 characteristic matrixes with less time by using the proposed approach.

Subsequently, we introduce the concept of the dynamic covering approximation space when deleting some objects and investigate the type-1 and type-2 characteristic matrixes of the dynamic covering when deleting some objects.

**Definition 4.9** Let  $(U, \mathcal{C})$  be a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ ,  $U^* \subseteq U$ ,  $U^- = U - U^*$ ,  $\mathcal{C}^- = \{C_1^-, C_2^-, \dots, C_m^-\}$ , and  $C_i^-$  is updated by deleting some objects of  $C_i$  which belong to  $U^*$ . Then  $(U^-, \mathcal{C}^-)$  is called the DO-covering approximation space of  $(U, \mathcal{C})$ .

In the sense of Definition 4.9,  $\mathcal{C}$  is called the original covering and  $\mathcal{C}^-$  is called the DO-covering of the original covering. Clearly, we can obtain the type-1 and type-2 characteristic matrixes of the DO-covering as the original covering.

We investigate that how to obtain the type-1 characteristic matrix  $\Gamma(\mathcal{C}^-)$  and the type-2 characteristic matrix  $\prod(\mathcal{C}^-)$  of the DO-covering. Suppose that we have obtained  $\Gamma(\mathcal{C}) = (b_{ij})_{n \times n}$  and  $\prod(\mathcal{C}) = (c_{ij})_{n \times n}$  shown in Section 3. If we delete the objects  $\{x_{i_k} | 1 \leq k \leq N\}$ , actually, it is easy to get  $\Gamma(\mathcal{C}^-)$  and  $\prod(\mathcal{C}^-)$  based on  $\Gamma(\mathcal{C})$  and  $\prod(\mathcal{C})$ , respectively. Concretely, we can obtain  $\Gamma(\mathcal{C}^-)$  by deleting the elements  $\{b_{i_k j}\}$ ,  $\{b_{j i_k}\}$ ,  $\{c_{i_k j}\}$  and  $\{c_{j i_k}\}$  ( $1 \leq j \leq n$ ,  $1 \leq k \leq N$ ) in  $\Gamma(\mathcal{C})$  and  $\prod(\mathcal{C})$ , respectively.

We employ an example to show that how to get the type-1 and type-2 characteristic matrixes of the DO-covering when deleting some objects.

**Example 4.10** (Continuation of Example 3.3) Suppose that we delete the object  $x_4$ , it is obvious that  $U^- = \{x_1, x_2, x_3\}$ ,  $\mathcal{C}^- = \{C_1^-, C_2^-, C_4^-\}$ , where  $C_1^- = \{x_1\}$ ,  $C_2^- = \{x_1, x_2\}$  and  $C_3^- = \{x_3\}$ . On one hand, we can delete the element  $b_{41}, b_{42}, b_{43}, b_{44}, b_{14}, b_{24}, b_{34}$  in the matrix  $\Gamma(\mathcal{C})$  shown in Example 3.3 and



$c_{41}, c_{42}, c_{43}, c_{44}, c_{14}, c_{24}, c_{34}$  in the matrix  $\prod(\mathcal{C})$  shown in Example 3.5. Then we have that

$$\begin{aligned}\Gamma(\mathcal{C}^-) &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\ \prod(\mathcal{C}^-) &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

On the other hand, we can get the above results as  $\Gamma(\mathcal{C})$  and  $\prod(\mathcal{C})$  shown in Examples 3.3 and 3.5, respectively:

$$\begin{aligned}\Gamma(\mathcal{C}^-) &= M_{\mathcal{C}^-} \cdot M_{\mathcal{C}^-}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\ \prod(\mathcal{C}^-) &= M_{\mathcal{C}^-} \odot M_{\mathcal{C}^-}^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

The time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $O((n-t)^2)$  if  $|U^*| = n-t$  and  $|\mathcal{C}| = m$ . But the time complexity is  $O(m * (n-t)^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix.

### 4.3 The characteristic matrixes of the dynamic covering when changing the attribute values of objects

In this subsection, we introduce the concept of the dynamic covering approximation space when changing the attribute values of some objects and investigate the type-1 and type-2 characteristic matrixes of the dynamic covering when there are changes of the attribute values for some objects.

**Definition 4.11** Let  $(U, \mathcal{C})$  be a covering approximation space, where  $U = \{x_1, x_2, \dots, x_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ . If we cancel some objects  $x$  from  $C \in \mathcal{C}$  and add them into other element of  $\mathcal{C}$ , and  $\mathcal{C}^* = \{C_1^*, C_2^*, \dots, C_m^*\}$ , where  $C_i^*$  is the updated version of  $C_i$  by deleting or adding objects. Then  $(U, \mathcal{C}^*)$  is called the CA-covering approximation space of  $(U, \mathcal{C})$ .

Generally speaking, if some attribute values are revised with time in the original approximation space  $(U, \mathcal{C})$ , then we will get the CA-covering approximation space  $(U, \mathcal{C}^*)$ . Maybe we get  $|\mathcal{C}| < |\mathcal{C}^*|$  if the change of attribute values of objects results in that the objects do not belong to the existing element of  $\mathcal{C}$ , and we will discuss the the above situation at the end of this subsection.

We focus on investigating the situation that  $|\mathcal{C}| = |\mathcal{C}^*|$ . Concretely, we only discuss that an object is deleted in an element of  $\mathcal{C}$  and added into another element of  $\mathcal{C}$ . Suppose that we have got  $\Gamma(\mathcal{C})$ , and the attribute value of  $x_k \in U$  is revised and it is deleted in  $C_I \in \mathcal{C}$  and added into  $C_J \in \mathcal{C}$ . We study the relationship between  $\Gamma(\mathcal{C})$  and  $\Gamma(\mathcal{C}^*)$  by showing them as follows. Concretely, we see that  $b_{ij} = c_{ij}$  for  $i \neq k$  and  $j \neq k$ . Since  $c_{ki} = c_{ik}$  in  $\Gamma(\mathcal{C}^*)$ , we only need to compute  $c_{kj}$  for  $1 \leq j \leq n$ .

[illegible]

$$= \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1I} & \cdot & \cdot & \cdot & c_{1J} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2I} & \cdot & \cdot & \cdot & c_{2J} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{k1} & c_{k2} & \cdot & \cdot & \cdot & c_{kI} & \cdot & \cdot & \cdot & c_{kJ} & \cdot & \cdot & \cdot & c_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdot & \cdot & \cdot & c_{nI} & \cdot & \cdot & \cdot & c_{nJ} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}.$$

An example is employed to illustrate the process of the computing of the type-1 characteristic matrix with the proposed approach.

**Example 4.12** (Continuation of Example 3.3) If we delete  $x_1$  in  $C_1$  and add it into  $C_3$ , then we get  $\mathcal{C}^* = \{C_1^*, C_2^*, C_3^*\}$ , where  $C_1^* = \{x_4\}$ ,  $C_2^* = \{x_1, x_2, x_4\}$  and  $C_3^* = \{x_1, x_3, x_4\}$ . Suppose that  $\Gamma(\mathcal{C}) = (b_{ij})_{4 \times 4}$  shown in Example 3.3 and  $\Gamma(\mathcal{C}^*) = (c_{ij})_{4 \times 4}$ . Actually, we have that  $b_{ij} = c_{ij}$  for  $2 \leq i, j \leq 4$  and only need to compute  $c_{11}, c_{12}, c_{13}$  and  $c_{14}$ .

$$\Gamma(\mathcal{C}^*) = M_{\mathcal{C}^*} \cdot M_{\mathcal{C}^*}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & 1 & 0 & 1 \\ c_{31} & 0 & 1 & 1 \\ c_{41} & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to get that  $c_{11} = c_{12} = c_{13} = c_{14} = 1$  and

$$\Gamma(\mathcal{C}^*) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & 1 & 0 & 1 \\ c_{31} & 0 & 1 & 1 \\ c_{41} & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then, we investigate the relationship  $\Pi(\mathcal{C})$  and  $\Pi(\mathcal{C}^*)$  which are shown as follows. Concretely, we have that  $b_{ij} = c_{ij}$  for  $i \neq k$  and  $j \neq k$ . Thus we only need to compute  $c_{ik}$  for  $1 \leq i \leq n$  and  $c_{kj}$  for  $1 \leq j \leq n$ .

$$\Pi(\mathcal{C}) = M_{\mathcal{C}} \odot M_{\mathcal{C}}^T = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1I} & \cdot & \cdot & \cdot & a_{1J} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2I} & \cdot & \cdot & \cdot & a_{2J} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & a_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nI} & \cdot & \cdot & \cdot & a_{nJ} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

$$\prod(\mathcal{C}^*) = M_{\mathcal{C}^*} \odot M_{\mathcal{C}^*}^T =$$

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**Example 4.13** (Continuation of Example 4.12) Suppose that  $\prod(\mathcal{C}) = (b_{ij})_{4 \times 4}$  shown in Example 3.3 and  $\prod(\mathcal{C}^*) = (c_{ij})_{4 \times 4}$ . Actually, we have that  $b_{ij} = c_{ij}$  for  $2 \leq i, j \leq 4$ . Thus we only need to compute  $c_{11}, c_{12}, c_{13}, c_{14}, c_{21}, c_{31}$  and  $c_{41}$ .

$$\prod(\mathcal{C}^*) = M_{\mathcal{C}^*} \odot M_{\mathcal{C}^*}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & 1 & 0 & 1 \\ c_{31} & 0 & 1 & 1 \\ c_{41} & 0 & 0 & 1 \end{bmatrix}.$$

Consequently, we get that  $c_{11} = c_{21} = c_{31} = c_{41} = 1, c_{12} = c_{13} = c_{14} = 0$ . Then we have that

$$\prod(\mathcal{C}^*) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Due to the change of the attribute value of the object  $x_k \in U$ , maybe we have that  $x$  does not belong to any element of  $\mathcal{C}$ . So we suppose that  $C_{m+1}^* = \{x_k\}$  and  $\mathcal{C}^* = \{C_1^*, C_2^*, \dots, C_m^*, C_{m+1}^*\}$ . We investigate the relationship  $M_{\mathcal{C}}$  and  $M_{\mathcal{C}^*}$  and suppose that  $\Gamma(\mathcal{C}) = (b_{ij})_{n \times n}$ ,  $\Gamma(\mathcal{C}^*) = (c_{ij})_{n \times n}$ ,  $\prod(\mathcal{C}) = (b_{ij}^*)_{n \times n}$  and  $\prod(\mathcal{C}^*) = (c_{ij}^*)_{n \times n}$ . Concretely, we have that  $b_{ij} = c_{ij}$  and  $b_{ij}^* = c_{ij}^*$  for  $i \neq k$  and  $j \neq k$ . Thus, we can obtain  $\Gamma(\mathcal{C}^*)$  and  $\prod(\mathcal{C}^*)$  by computing  $c_{ik}$  ( $1 \leq i \leq n$ ),  $c_{ik}^*$  ( $1 \leq i \leq n$ ) and  $c_{kj}^*$  ( $1 \leq j \leq n$ ).

The time complexity of computing the type-1 (respectively, type-2) characteristic matrix is  $O(n) + O(n^2)$  if  $|U^*| = n$ ,  $|\mathcal{C}| = m$  and an attribute value is revised. But the time complexity is  $O(m * n^2)$  by using the concept of the type-1 (respectively, type-2) characteristic matrix. Therefore, we can get the type-1 and type-2 characteristic matrixes with less time by using the proposed approach.

## 5 Conclusions

The approximations of concepts and feature selections in the dynamic information system are important works of the rough set theory, and they are challenging issues in the field of the artificial intelligence. In this paper, we have introduced two approaches to constructing the approximations of concepts in the covering approximation space. After that, we have constructed the characteristic matrixes of five types of the dynamic coverings with the proposed approach. Additionally, we have employed several examples to illustrate the process of computing the characteristic matrixes of the dynamic coverings by using an incremental approach.

In the future, we will propose more effective approaches to constructing the characteristic matrixes of the covering. Additionally, we will focus on the development of effective approaches for knowledge discovery of dynamic information systems.

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